International Journal of Theoretical Physics, Vol. 45, No. 6, June 2006 (© 2006)
DOI: 10.1007/s10773-006-9100-9

Superposition in Nonlinear Wave and Evolution Equations

H. W. Schürmann,[1] V. S. Serov,[2] and J. Nickel[1,3]

Received 22 August 2005; accepted 30 January 2006
Published Online: May 16, 2006

Real and bounded elliptic solutions suitable for applying the Khare-Sukhatme superposition procedure are presented and used to generate superposition solutions of the generalized modified Kadomtsev-Petviashvili equation (gmKPE) and the nonlinear cubic-quintic Schrödinger equation (NLCQSE).

KEY WORDS: Linear superposition; solitary wave solution.

PACS: 02.30.Jr; 02.30.Gp

1. INTRODUCTION

As has been shown recently (Cooper *et al.*, 2002; Khare *et al.*, 2002a,b, 2003) (periodic) Jacobian elliptic functions (if they are solutions of a certain nonlinear wave and evolution equation (NLWEE)) are start solutions for generating new solutions of the NLWEE by a linear superposition procedure. Thus, elliptic functions are of specific importance for finding solutions of NLWEEs. On the other hand, based on a symmetry reduction method, a technique to obtain elliptic solutions of certain NLWEEs was proposed and applied to the gmKPE and the NLCQSE (Schürmann, 1996; Schürmann *et al.*, 2004a; Schürmann *et al.*, 2004b). It is the aim of the present paper to combine these approaches in order to obtain general elliptic solutions that can serve as start solutions for superposition ("suitable solutions").

The superposition procedure can be described as follows (Cooper *et al.*, 2002): If a solution of a NLWEE$[\psi(x, y, t)] = 0$ can be expressed in terms of

[1] Department of Physics, University of Osnabrück, D-49069 Osnabrück, Germany.
[2] Department of Mathematical Sciences, University of Oulu, FIN-90014, Finland.
[3] To whom correspondence should be addressed at Department of Physics, University of Osnabrück, D-49069 Osnabrück, Germany; e-mail: jnickel@uos.de.

Jacobian elliptic functions

$$\Psi(x, y, t) = \sum_{\nu=0}^{l} a_\nu \mathrm{qn}^\nu \left[\mu(x + ky + vt), m\right], \tag{1}$$

where qn is anyone of the Jacobian elliptic functions and a_ν, μ, c are constants, then the superposition solution (Cooper *et al.*, 2002, Eqs. (4), (14))

$$\widetilde{\Psi}(x, y, t) = \sum_{\lambda=1}^{p} \sum_{\nu=0}^{l} a_\nu \mathrm{qn}^\nu \left[\mu(x + ky + v_p t) + \frac{n(\lambda - 1)K(m)}{p}, m\right], \tag{2}$$

where $n \in \{2, 4\}$ (depending on the periodicity of the Jacobian elliptic function and on ν) and $K(m), m$ denote the complete elliptic integral of first kind and the modulus parameter ($0 \leq m \leq 1$), respectively, also may be a solution of the NLWEE. The number p is integer (it depends on the NLWEE whether it is even or/and odd) and the speed v_p can be determined by using certain remarkable, recently established, identities involving Jacobian elliptic functions (Khare *et al.*, 2002a, 2003). It should be noted, that the existence of solutions (1) of a certain NLWEE does not necessarily imply the existence of a solution (2). As shown in Cooper *et al.* (2002) and Khare *et al.* (2002b) solutions (2) exist for the Korteweg-de Vries equation (KdV), the Kadomtsev-Petviashvili equation (KP), the nonlinear (cubic) Schrödinger equation (NLSE), the $\lambda\phi^4$—field equation, the Sine-Gordon equation and the Boussinesq equation. On the other hand, it may happen, as will be seen below, that a solution (1) is known but does not lead to a solution (2). It is crucial for the procedure, that appropriate relations between Jacobian elliptic functions are known.

The symmetry reduction approach can be outlined as follows (Schürmann *et al.*, 2004b): The NLWEE$[\psi(x, t)] = 0$ is reduced by an appropriate transformation $\psi \to f$ (e. g., $\psi(x, t) = f(z), z = x - ct$), where f is supposed to obey the ordinary nonlinear differential equation ("basic equation")

$$\left(\frac{df(z)}{dz}\right)^2 = \alpha f^4 + 4\beta f^3 + 6\gamma f^2 + 4\delta f + \epsilon \equiv R(f), \tag{3}$$

(with real $z, f(z), \alpha, \beta, \gamma, \delta, \epsilon$), leading to an equation $P(f) = 0$, where P denotes a polynomial in f. Vanishing coefficients in the polynomial equation $P(f) = 0$ imply equations which partly determine the coefficients α, β, γ, δ, ϵ in Equation (3). In general, the coefficients depend on the structure and parameters of the NLWEE and, finally, on the parameters of the transformation $\psi \to f$. Thus, the problem of finding a solution of the NLWEE is reduced to finding an appropriate transformation that leads to the basic equation (3).

As is well known (Weierstrass, 1915; Whittaker *et al.*, 1927) the solution $f(z)$ of Eq. (3) can be written as

$$f(z) = f_0 + \frac{R'(f_0)}{4[\wp(z; g_2, g_3) - \frac{1}{24}R''(f_0)]}, \tag{4}$$

where f_0 is a simple root of $R(f)$[4] and the prime denotes differentiation with respect to f.

The invariants g_2, g_3 of Weierstrass' elliptic function $\wp(z; g_2, g_3)$ are related to the coefficients of $R(f)$ by (Chandrasekharan, 1985)

$$g_2 = \alpha\epsilon - 4\beta\delta + 3\gamma^2, \tag{5}$$

$$g_3 = \alpha\gamma\epsilon + 2\beta\gamma\delta - \alpha\delta^2 - \gamma^3 - \epsilon\beta^2. \tag{6}$$

The discriminant (of $\wp$ and R (Chandrasekharan, 1985))

$$\triangle = g_2^3 - 27g_3^2, \tag{7}$$

is suitable to classify the behaviour of $f(z)$. The conditions

$$\triangle \neq 0 \quad \text{or} \quad \triangle = 0, \quad g_2 > 0, \quad g_3 > 0 \tag{8}$$

lead to periodic solutions (Schürmann *et al.*, 2004b), whereas the conditions (Abramowitz *et al.*, 1972)

$$\triangle = 0, \quad g_2 \geq 0, \quad g_3 \leq 0 \tag{9}$$

are associated with solitary wave like solutions.

Physical solutions $f(z)$ must be real and bounded. Considering the phase diagram of $R(f)$ (Schürmann, 1996; Drazin, 1983) one obtains conditions, expressed in terms of the coefficients of the basic equation, that determine physical solutions. For convenience these conditions are referred to as the phase diagram conditions (PDC) in the following.

2. ELLIPTIC START SOLUTIONS FOR SUPERPOSITION

To apply the superposition procedure it is important to know whether a solution of the NLWEE according to (1) exists. To check this it is useful to rewrite

[4] The general solution of Eq. (3) reads (Weierstrass, 1915; Whittaker *et al.*, 1927)

$$f(z) = f_0 + \frac{\sqrt{R(f_0)}\frac{d\wp(z;g_2,g_3)}{dz} + \frac{1}{2}R'(f_0)[\wp(z; g_2, g_3) - \frac{1}{24}R''(f_0)] + \frac{1}{24}R(f_0)R'''(f_0)}{2[\wp(z; g_2, g_3) - \frac{1}{24}R''(f_0)]^2 - \frac{1}{48}R(f_0)R''''(f_0)}.$$

Weierstrass' function $\wp$ as[5]

$$\wp(z) = e_3 + \frac{e_1 - e_3}{\mathrm{sn}^2(\sqrt{e_1 - e_3}z, m)}, \tag{10}$$

where $e_1 \geq e_2 \geq e_3$ denote the roots of the equation

$$4s^3 - g_2 s - g_3 = 0. \tag{11}$$

Substitution of Eq. (10) into Eq. (4) yields[6]

$$f(z) = \frac{(\alpha f_0{}^3 + 4\beta f_0{}^2 + 2e_3 f_0 + 5\gamma f_0 + 2\delta)\mathrm{sn}^2(\sqrt{e_1 - e_3}z, m) + 2(e_1 - e_3) f_0}{(-\alpha f_0{}^2 - 2\beta f_0 + 2e_3 - \gamma)\mathrm{sn}^2(\sqrt{e_1 - e_3}z, m) + 2(e_1 - e_3)}, \tag{12}$$

with $m = \frac{e_2 - e_3}{e_1 - e_3}$. Comparison with Eq. (1) shows that

$$-\alpha f_0{}^2 - 2\beta f_0 + 2e_3 - \gamma = 0 \tag{13}$$

is a necessary and sufficient condition that defines the subset of solutions (1). If $\alpha = 0$ holds the simple root f_0 of $R(f)$ can be choosen such that Eq. (13) and PDC are satisfied.[7] If $\alpha \neq 0$ and $\beta = \delta = 0$ Eq. (13) is satisfied also. If $\alpha \neq 0$ and $\beta \neq 0$, $\delta = \epsilon = 0$, Eq. (13), PDC, and the condition $\triangle = 0$, $g_3 > 0$ are not consistent, so that trigonometric functions (which are possible for $\triangle = 0$, $g_3 > 0$) are not suitable for superposition, because $f(z)$ is a constant according to the general solution of Eq. (3).[4]

Equation (13) represents a relation between the parameters $\{\alpha, \beta, \gamma, \delta, \epsilon\}$ and thus determines a subset of parameters of the problem modelled by the NLWEE for which further solutions can be generated by superposition according to Eq. (2). Combining Eqs. (12) and (13) (with $\alpha = 0$) we obtain

$$f(z) = \frac{2e_3 - \gamma}{2\beta} + \frac{12e_3^2 - 3\gamma^2 + 4\beta\delta}{4\beta(e_1 - e_3)}\mathrm{sn}^2(\sqrt{e_1 - e_3}z, m), \tag{14}$$

where e_1, e_3 must be chosen as the largest and smallest roots of Eq. (11), respectively, so that the condition (13) is valid for a simple root f_0 of Eq. (3) that satisfies the PDC.

[5] We assumed $\triangle > 0$. If $\triangle < 0$, substitution of $\wp(z)$ according to Abramowitz *et al.* (1972), 18.9.11 does not lead to an expression of form (1). Furthermore, as will be seen below, we obtain a polynomial $R(f)$ or $R(h)$, $h = f^2$, of third degree for generating new solutions by linear superposition. But the PDC is not satisfied for a third-degree polynomial with $\triangle < 0$, because two of the three roots are complex conjugate.

[6] Since we are interested in physical periodic solutions we can always assume that a simple root exists.

[7] If $\beta = 0$ holds γ must be negative otherwise Eq. (13) and PDC are not fulfilled. For f_0 to be a simple root $\delta^2 - \frac{3}{2}\epsilon\gamma$ must be positive. If $\beta \neq 0$ the discriminant $\triangle$ does not vanish, so that $f_0 = \frac{2e_3 - \gamma}{2\beta}$ is a simple root of $R(f)$.

Equation (14) can be evaluated explicitely subject to the two cases $\alpha = 0$ and $\alpha \neq 0$, $\beta = \delta = 0$, respectively. If $\alpha = 0$ and, for simplicity, $\epsilon = 0$ the start solutions for superposition are

$$f(z) = \begin{cases} -\dfrac{3\gamma + \sqrt{9\gamma^2 - 16\beta\delta}}{4\beta}\mathrm{dn}^2 \\ \times\left(\dfrac{1}{2}\sqrt{3\gamma + \sqrt{9\gamma^2 - 16\beta\delta}}z, \dfrac{2\sqrt{9\gamma^2 - 16\beta\delta}}{3\gamma + \sqrt{9\gamma^2 - 16\beta\delta}}\right), & \beta\delta > 0,\ \gamma > 0, \\ \dfrac{4\delta}{-3\gamma + \sqrt{9\gamma^2 - 16\beta\delta}}\mathrm{sn}^2 \\ \times\left(\dfrac{1}{2}\sqrt{-3\gamma + \sqrt{9\gamma^2 - 16\beta\delta}}z, \dfrac{3\gamma + \sqrt{9\gamma^2 - 16\beta\delta}}{3\gamma - \sqrt{9\gamma^2 - 16\beta\delta}}\right), & \beta\delta > 0, \gamma < 0, \\ -\dfrac{3\gamma + \sqrt{9\gamma^2 - 16\beta\delta}}{4\beta}\mathrm{cn}^2 \\ \times\left(\dfrac{(9\gamma^2 - 16\beta\delta)^{\frac{1}{4}}}{\sqrt{2}}z, \dfrac{3\gamma + \sqrt{9\gamma^2 - 16\beta\delta}}{2\sqrt{9\gamma^2 - 16\beta\delta}}\right), & \beta\delta < 0, \end{cases} \tag{15}$$

where the various possibilities to satisfy (11) and (13) have been taken into account and $\triangle = 4\beta^2\delta^2(9\gamma^2 - 16\beta\delta) > 0$ is necessary and sufficient to fulfill PDC (Bronstein *et al.*, 2000).

If $\alpha \neq 0$, $\beta = \delta = 0$ the start solutions read

$$h(z) = \begin{cases} -\dfrac{3\gamma + \sqrt{9\gamma^2 - \alpha\epsilon}}{\alpha}\mathrm{dn}^2 \\ \times\left(\sqrt{3\gamma + \sqrt{9\gamma^2 - \alpha\epsilon}}z, \dfrac{2\sqrt{9\gamma^2 - \alpha\epsilon}}{3\gamma + \sqrt{9\gamma^2 - \alpha\epsilon}}\right), & \alpha < 0,\ \gamma > 0,\ \epsilon < 0, \\ \dfrac{\epsilon}{-3\gamma + \sqrt{9\gamma^2 - \alpha\epsilon}}\mathrm{sn}^2 \\ \times\left(\sqrt{-3\gamma + \sqrt{9\gamma^2 - \alpha\epsilon}}z, \dfrac{3\gamma + \sqrt{9\gamma^2 - \alpha\epsilon}}{3\gamma - \sqrt{9\gamma^2 - \alpha\epsilon}}\right), & \alpha > 0,\ \gamma < 0,\ \epsilon > 0, \\ -\dfrac{3\gamma + \sqrt{9\gamma^2 - \alpha\epsilon}}{\alpha}\mathrm{cn}^2 \\ \times\left(\sqrt{2}(9\gamma^2 - \alpha\epsilon)^{\frac{1}{4}}z, \dfrac{3\gamma + \sqrt{9\gamma^2 - \alpha\epsilon}}{2\sqrt{9\gamma^2 - \alpha\epsilon}}\right), & \alpha < 0,\ \epsilon > 0, \end{cases} \tag{16}$$

where $\triangle = 64\alpha^2\epsilon^2(9\gamma^2 - \alpha\epsilon) > 0$ and - according to the Cartesian sign rule - three numbers of sign reversals in the sequence of coefficients of $R(h)$ or $\triangle > 0$ and $\alpha > 0$ and two sign reversals to fulfill PDC.

To sum up, Eqs. (15) and (16) represent all elliptic solutions with $\alpha = 0$, $\epsilon = 0$ and $\alpha \neq 0$, $\beta = \delta = 0$, respectively, that are suitable for the the procedure suggested by Khare and Sukhatme. "All elliptic" means that the solutions presented in Cooper *et al.* (2002) and Khare *et al.* (2002b) are particular cases of Eqs. (15) and (16). "Suitable" includes that the superposition procedure may fail if solutions according to Eq. (15) or (16) are inserted into the NLWEE in question leading to conditions that cannot be evaluated with respect to v_p (cf. Eq. (2)) because the associated relations between Jacobian functions are unknown (cf. Section 3.). Examples to obtain superposition solutions are presented in the following. Equation (14) can be evaluated in the same manner subject to the PDC to obtain physical elliptic solutions if the simplifying assumption $\epsilon = 0$ does not hold.

3. SUPERPOSITION SOLUTIONS OF THE GENERALIZED MODIFIED KADOMTSEV-PETVIASHVILI EQUATION

The approach outlined in the previous section can be elucidated by investigation of the gmKPE (Superposition solutions of the NLCQSE are presented in A.)

$$\psi_{xt} + ((a + b\psi^q)\psi^q\psi_x)_x + c\psi_{xxxx} - \sigma^2\psi_{yy} = 0, \tag{17}$$

where a, b, c, q, σ^2 are real constants. As shown previously (Schürmann *et al.*, 2004a) elliptic traveling-wave solutions to Eq. (17) exist. The set of these solutions is determined by

$$\begin{aligned} \psi(x, y, t) &= f(z)^{1/q}, \quad q \neq 0, \\ z &= x + ky + vt, \\ f_z^2 &= \alpha f^4 + 4\beta f^3 + 6\gamma f^2 + 4\delta f + \epsilon, \end{aligned} \tag{18}$$

where $\alpha, \beta, \gamma, \delta, \epsilon$ are given by Eqs. (16a)–(16g) in Schürmann *et al.* (2004a). As shown above, the conditions $\alpha = 0$ or $\beta = \delta = 0, \alpha \neq 0$ lead to suitable solutions. Imposing additionally the PDC and condition (13), respectively, the parameters of solutions (18) are

$$q = \frac{1}{2}, \quad \alpha = -\frac{b}{12c}, \quad \beta = 0, \quad \gamma = \frac{k^2\sigma^2 - v}{24c}, \quad \delta = 0, \quad \epsilon \neq 0, \quad c \neq 0, \tag{19}$$

$$q = 1, \quad \alpha = 0, \quad \beta = -\frac{a}{12c}, \quad \gamma = \frac{k^2\sigma^2 - v}{6c}, \quad \epsilon = 0, \quad c \neq 0, \tag{20}$$

$$q = 1, \quad \alpha = -\frac{b}{6c}, \quad \beta = 0, \quad \gamma = \frac{k^2\sigma^2 - v}{6c}, \quad \delta = 0, \quad \epsilon \neq 0, \quad c \neq 0, \tag{21}$$

$$q = 2, \quad \alpha = 0, \quad \beta = -\frac{a}{6c}, \quad \gamma = \frac{2(k^2\sigma^2 - v)}{3c}, \quad \epsilon = 0, \quad c \neq 0. \tag{22}$$

Referring to (19) and (20) first, solutions according to (16) and (15), respectively, have to be evaluated. Inserting (19) into (16), one obtains the suitable start solutions

$$\psi(x, y, t) = \begin{cases} B_1 \,\mathrm{dn}^2\left[\mu_1(x + ky + vt), m_1\right], \ \dfrac{b}{c} > 0, \ \dfrac{k^2\sigma^2 - v}{c} > 0, \ \epsilon < 0, \\[2ex] B_2 \,\mathrm{sn}^2\left[\mu_2(x + ky + vt), m_2\right], \ \dfrac{b}{c} < 0, \ \dfrac{k^2\sigma^2 - v}{c} < 0, \ \epsilon > 0, \\[2ex] B_3 \,\mathrm{cn}^2\left[\mu_3(x + ky + vt), m_3\right], \ \dfrac{b}{c} > 0, \ \epsilon > 0, \end{cases} \tag{23}$$

where B_j, μ_j, m_j are determined by inserting the parameters α, γ, ϵ according to (19) into (16). Formally the same result is obtained by inserting (20) into (15).

Referring, secondly, to (21) the solutions follow from (16) as

$$\psi(x, y, t) = \begin{cases} B_1 \,\mathrm{dn}\left[\mu_1(x + ky + vt), m_1\right], \ \dfrac{b}{c} > 0, \ \dfrac{k^2\sigma^2 - v}{c} > 0, \ \epsilon < 0, \\[2ex] B_2 \,\mathrm{sn}\left[\mu_2(x + ky + vt), m_2\right], \ \dfrac{b}{c} < 0, \ \dfrac{k^2\sigma^2 - v}{c} < 0, \ \epsilon > 0, \\[2ex] B_3 \,\mathrm{cn}\left[\mu_3(x + ky + vt), m_3\right], \ \dfrac{b}{c} > 0, \ \epsilon > 0, \end{cases} \tag{24}$$

where (again) B_j, μ_j, m_j are determined from (16) with parameters according to (21). Formally the same results are given by (22) and (15).

According to Eq. (2) the first solution in (24) leads to a superposition solution for $p = 2$

$$\begin{gathered} \widetilde{\psi}(x, y, t) = B \sum_{i=1}^{2} \mathrm{dn}\left(\mu(x + ky + v_2 t) + (i - 1)K(m), m\right), \\ B = B_1, \mu = \mu_1, m = m_1. \end{gathered} \tag{25}$$

Inserting $\widetilde{\psi}(x, y, t)$ (denoting $\mathrm{d}_i = \mathrm{dn}\,(\mu(x + ky + v_2 t) + (i - 1)K(m), m))$ into Eq. (17) ($a = 0$, because $\beta = 0$ according to (21)) we get

$$
\begin{aligned}
&(-Bm\mu v_2 - Bc\mu^3(2m - m^2))\frac{d}{dx}\sum_{i=1}^{2}\mathrm{s}_i\mathrm{c}_i + \sigma^2 Bm\mu k\frac{d}{dy}\sum_{i=1}^{2}\mathrm{s}_i\mathrm{c}_i \\
&\quad + 2bB^3m^2\mu^2\sum_{i=1}^{2}\mathrm{d}_i\left(\sum_{i=1}^{2}\mathrm{s}_i\mathrm{c}_i\right)^2 - m\mu bB^3\left(\sum_{i=1}^{2}\mathrm{d}_i\right)^2\frac{d}{dx}\sum_{i=1}^{2}\mathrm{s}_i\mathrm{c}_i \\
&\quad + 6Bcm\mu^3\frac{d}{dx}\sum_{i=1}^{2}\mathrm{d}_i^2\mathrm{s}_i\mathrm{c}_i = 0. \qquad (26)
\end{aligned}
$$

The last three terms of Eq. (26) can be simplified as follows.

Using $\mathrm{d}_1\mathrm{d}_2 = \sqrt{1-m}$ and $\mathrm{c}_1\mathrm{s}_1\mathrm{d}_2 + \mathrm{c}_2\mathrm{s}_2\mathrm{d}_1 = 0$ (Khare *et al.*, 2002a, Eqs. (31), (39)) we obtain

$$
\left(\sum_{i=1}^{2}\mathrm{d}_i\right)^2\sum_{i=1}^{2}\mathrm{s}_i\mathrm{c}_i = \sum_{i=1}^{2}\mathrm{d}_i^2\mathrm{s}_i\mathrm{c}_i + \sqrt{1-m}\sum_{i=1}^{2}\mathrm{s}_i\mathrm{c}_i. \qquad (27)
$$

Evaluating $\frac{d}{dx}((\sum_{i=1}^{2}\mathrm{d}_i)^2\sum_{i=1}^{2}\mathrm{s}_i\mathrm{c}_i)$ and using Eq. (27), Eq. (26) can be rewritten as

$$
\begin{aligned}
&(-Bm\mu v_2 - Bc\mu^3(2m - m^2) - m\mu bB^3\sqrt{1-m})\frac{d}{dx}\sum_{i=1}^{2}\mathrm{s}_i\mathrm{c}_i \\
&\quad + \sigma^2 Bm\mu k\frac{d}{dy}\sum_{i=1}^{2}\mathrm{s}_i\mathrm{c}_i + Bm\mu\left(6c\mu^2 - bB^2\right)\frac{d}{dx}\sum_{i=1}^{2}\mathrm{d}_i^2\mathrm{s}_i\mathrm{c}_i = 0. \qquad (28)
\end{aligned}
$$

The expression $(6c\mu^2 - bB^2)$ vanishes identically.[8] With $\frac{d}{dy}\sum_{i=1}^{2}\mathrm{s}_i\mathrm{c}_i = k\frac{d}{dx}\sum_{i=1}^{2}\mathrm{s}_i\mathrm{c}_i$ Eq. (28) reads

$$
\begin{aligned}
&(-Bm\mu v_2 - Bc\mu^3(2m - m^2) - m\mu bB^3\sqrt{1-m} + \sigma^2 Bm\mu k^2) \\
&\quad \times\frac{d}{dx}\sum_{i=1}^{2}\mathrm{s}_i\mathrm{c}_i = 0, \qquad (29)
\end{aligned}
$$

so that the speed v_2 is given by

$$
v_2 = -c\mu^2(2 - m) - bB^2\sqrt{1-m} + \sigma^2k^2. \qquad (30)
$$

[8] If parameters according to Eqs. (21) are inserted into Eq. (16) one obtains B, μ, m so that $6Bcm\mu^3 - m\mu bB^3$ in Eq. (28) vanishes identically.

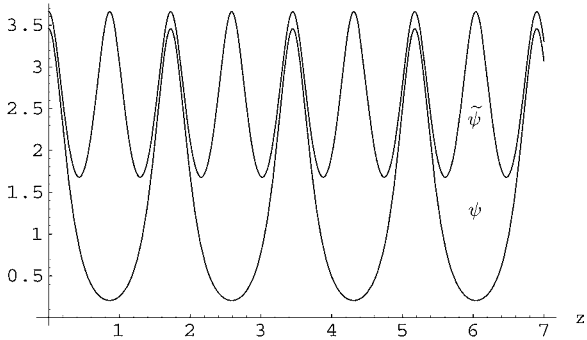

Fig. 1. The start solution $\psi(z)$ (cf. first solution of Eq. (24)) and the superposition solution $\tilde{\psi}(z)$ (cf. Eq. (25)) for $\alpha = -2$, $\gamma = 4$, $\epsilon = -1$, $z = x + ky + vt$ and $z = x + ky + v_2 t$, respectively.

Thus, we have found a superposition solution of Eq. (17) for this particular case.

The start solution and the superposition solution are shown in Fig. 1.

We can generate superposition solutions for $p = 3$ from (23). As an example we consider the solution of the form dn^2 in detail and compare it with the results of Cooper *et al.* (2002). According to Eq. (2) the superposition ansatz reads

$$\tilde{\psi}(x, y, t) = B \sum_{i=1}^{3} \mathrm{dn}^2 \left(\mu(x + ky + v_3 t) + \frac{2(i-1)K(m)}{3}, m \right),$$
$$B = B_1, \mu = \mu_1, m = m_1. \tag{31}$$

Inserting $\tilde{\psi}(x, y, t)$ (denoting $\mathrm{d}_i = \mathrm{dn}(\mu(x + ky + v_3 t) + \frac{2(i-1)K(m)}{3}, m))$ into Eq. (17), we obtain

$$2Bm\mu(v_3 + 8c\mu^2 - 4cm\mu^2)\frac{d}{dx}\sum_{i=1}^{3} \mathrm{d}_i \mathrm{s}_i \mathrm{c}_i + 2\sigma^2 Bm\mu k \frac{d}{dy}\sum_{i=1}^{3} \mathrm{d}_i \mathrm{s}_i \mathrm{c}_i$$
$$+ 4B^2 bm^2\mu^2 \left(\sum_{i=1}^{3} \mathrm{d}_i \mathrm{s}_i \mathrm{c}_i\right)^2 - 2B^2 bm\mu \sum_{i=1}^{3} \mathrm{d}_i^2 \frac{d}{dx} \sum_{i=1}^{3} \mathrm{d}_i \mathrm{s}_i \mathrm{c}_i$$
$$+ 24Bcm\mu^3 \frac{d}{dx}\sum_{i=1}^{3} \mathrm{d}_i^3 \mathrm{s}_i \mathrm{c}_i = 0. \tag{32}$$

The last three terms can be rewritten as

$$-2B^2bm\mu\left(-2m\mu\left(\sum_{i=1}^{3}\mathrm{d}_i\mathrm{s}_i\mathrm{c}_i\right)^2+\sum_{i=1}^{3}\mathrm{d}_i^2\frac{d}{dx}\sum_{i=1}^{3}\mathrm{d}_i\mathrm{s}_i\mathrm{c}_i\right.$$

$$\left.-12\frac{c\mu^2}{bB}\frac{d}{dx}\sum_{i=1}^{3}\mathrm{d}_i^3\mathrm{s}_i\mathrm{c}_i\right), \tag{33}$$

whereas evaluation of $\frac{d}{dx}(\sum_{i=1}^{3}\mathrm{d}_i^2\sum_{j\neq i}\mathrm{d}_j\mathrm{s}_j\mathrm{c}_j)$ yields

$$\begin{aligned}\frac{d}{dx}\left(\sum_{i=1}^{3}\mathrm{d}_i^2\sum_{j\neq i}\mathrm{d}_j\mathrm{s}_j\mathrm{c}_j\right) &= -2m\mu\left(\sum_{i=1}^{3}\mathrm{d}_i\mathrm{s}_i\mathrm{c}_i\right)^2+\sum_{i=1}^{3}\mathrm{d}_i^2\frac{d}{dx}\sum_{i=1}^{3}\mathrm{d}_i\mathrm{s}_i\mathrm{c}_i\\ &\quad+2m\mu\sum_{i=1}^{3}\mathrm{d}_i^2\mathrm{s}_i^2\mathrm{c}_i^2-\sum_{i=1}^{3}\left(\mathrm{d}_i^2\frac{d}{dx}\mathrm{d}_i\mathrm{s}_i\mathrm{c}_i\right)\\ &= -2m\mu\left(\sum_{i=1}^{3}\mathrm{d}_i\mathrm{s}_i\mathrm{c}_i\right)^2+\sum_{i=1}^{3}\mathrm{d}_i^2\frac{d}{dx}\\ &\quad\times\sum_{i=1}^{3}\mathrm{d}_i\mathrm{s}_i\mathrm{c}_i-\frac{d}{dx}\sum_{i=1}^{3}\mathrm{d}_i^3\mathrm{s}_i\mathrm{c}_i.\end{aligned} \tag{34}$$

Because $12\frac{c\mu^2}{bB}=1$ (in Eq. (33)) holds identically, we can use Eq. (34) and (Khare *et al.*, 2002b, Eq. (11))

$$\frac{d}{dx}\left(\sum_{i=1}^{3}\mathrm{d}_i^2\sum_{j\neq i}\mathrm{d}_j\mathrm{s}_j\mathrm{c}_j\right)=A(3,m)\frac{d}{dx}\sum_{i=1}^{3}\mathrm{d}_i\mathrm{s}_i\mathrm{c}_i, \tag{35}$$

to rewrite Eq. (32) as

$$\begin{aligned}&-2Bm\mu(v_3+8c\mu^2-4cm\mu^2+BbA(3,m))\frac{d}{dx}\sum_{i=1}^{3}\mathrm{d}_i\mathrm{s}_i\mathrm{c}_i\\ &\quad+2\sigma^2Bm\mu k\frac{d}{dy}\sum_{i=1}^{3}\mathrm{d}_i\mathrm{s}_i\mathrm{c}_i=0.\end{aligned} \tag{36}$$

Using $\frac{d}{dy}\sum_{i=1}^{3}\mathrm{d}_i\mathrm{s}_i\mathrm{c}_i=k\frac{d}{dx}\sum_{i=1}^{3}\mathrm{d}_i\mathrm{s}_i\mathrm{c}_i$ this equation reads

$$-2Bm\mu(v_3+8c\mu^2-4cm\mu^2-\sigma^2k^2+BbA(3,m))\frac{d}{dx}\sum_{i=1}^{3}\mathrm{d}_i\mathrm{s}_i\mathrm{c}_i=0. \tag{37}$$

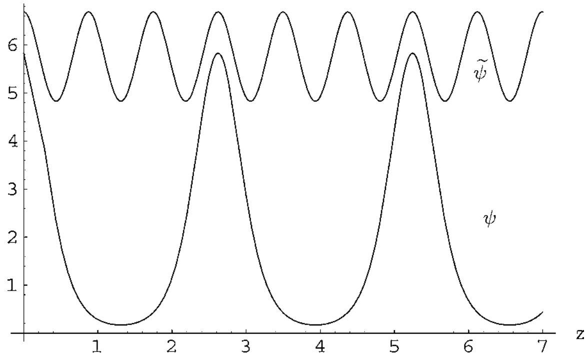

Fig. 2. The start solution $\psi(z)$ (cf. first solution of Eq. (23)) and the superposition solution $\widetilde{\psi}(z)$ (cf. Eq. (31)) for $\alpha = -1$, $\gamma = 1$, $\epsilon = -1$, $z = x + ky + vt$ and $z = x + ky + v_3 t$, respectively.

Thus, the speed v_3 in the superposition solution (31) (of a particular case) of Eq. (17) is given by

$$v_3 = 4cm\mu^2 + \sigma^2 k^2 - 8c\mu^2 - BbA(3, m). \tag{38}$$

The start solution and the superposition solution are shown in Fig. 2.

Applying an analogous procedure with the ansatz

$$\begin{aligned} \widetilde{\psi}(x, y, t) &= B \sum_{i=1}^{3} \mathrm{sn}^2 \left(\mu(x + ky + v_3 t) + \frac{2(i-1)K(m)}{3}, m \right), \\ B &= B_2,\ \mu = \mu_2,\ m = m_2 \end{aligned} \tag{39}$$

and with the ansatz

$$\begin{aligned} \widetilde{\psi}(x, y, t) &= B \sum_{i=1}^{3} \mathrm{cn}^2 \left(\mu(x + ky + v_3 t) + \frac{2(i-1)K(m)}{3}, m \right), \\ B &= B_3,\ \mu = \mu_3,\ m = m_3 \end{aligned} \tag{40}$$

we obtain superposition solutions with

$$v_3 = 4cm\mu^2 + 4c\mu^2 + \sigma^2 k^2 + Bb\frac{A(3, m) - 2}{m} \tag{41}$$

for solution (39) and

$$v_3 = -8cm\mu^2 + 4c\mu^2 + \sigma^2 k^2 - Bb\frac{A(3, m) - 2(1 - m)}{m} \tag{42}$$

for solution (40).

In deriving (41) and (42) we have used the relations

$$\frac{d}{dx}\left(\sum_{i=1}^{3} \mathrm{s}_i^2 \sum_{j\neq i}^{3} \mathrm{s}_j \mathrm{c}_j \mathrm{d}_j\right) = -\frac{1}{m}\left(A(3,m)-2\right)\frac{d}{dx}\sum_{i=1}^{3} \mathrm{s}_i \mathrm{c}_i \mathrm{d}_i \tag{43}$$

and

$$\frac{d}{dx}\left(\sum_{i=1}^{3} \mathrm{c}_i^2 \sum_{j\neq i}^{3} \mathrm{s}_j \mathrm{c}_j \mathrm{d}_j\right) = \frac{1}{m}\left(A(3,m)-2(1-m)\right)\frac{d}{dx}\sum_{i=1}^{3} \mathrm{s}_i \mathrm{c}_i \mathrm{d}_i, \tag{44}$$

respectively, which follow from Eq. (35) and well known relations between Jacobian elliptic functions.

Comparison of the Kadomtsev-Petivashvili equation together with the ansatz considered by Cooper, Khare and Sukhatme (Cooper *et al.*, 2002, Eqs. (1),(4))with the Eqs. (17), (19) and (31) shows that, apart from an additive constant (Jaworski *et al.*, 2003), our result (38) is consistent with that of (Cooper *et al.*, 2002, Eq. (11), $\beta = 0$) .The cases related to (41), (42) have not been considered in Cooper *et al.* (2002).

To conclude, we note that real and bounded suitable solutions of the gmKPE only exist for four different values of q (cf. (19)–(22)), though there is no restriction for q (apart from being real) of the known elliptic solutions of the gmKPE (Schürmann *et al.*, 2004a).

The second of Eqs. (24) does not lead to a superposition solution although the solution has the form (1).[9] In this case, it seems that an appropriate identity for Jacobian elliptic functions does not exist. Thus, the claim at the end of Cooper *et al.* (2002) seems to strong.

4. SUMMARY AND CONCLUDING REMARKS

By combining the superposition principle and symmetry reduction we obtained general elliptic solutions suitable for superposition. The results were applied to the gmKP Eand the NLCQSE (see Appendix). In Cooper *et al.* (2002) particular elliptic solutions for generating superposition solutions of the NLSE and the KPE were used. As outlined above we start from (general) suitable solutions (cf. Eqs. (15), (16), (23), (24)) to obtain superposition solutions more general than those in Cooper *et al.* (2002). We note that there are no restrictions in advance for

[9] An ansatz $\widetilde{\psi} = \mathrm{cn}$ or $\widetilde{\psi} = \mathrm{sn}$ (cf. Eq. (25)) leads to equations which have the form (26). Because there is no relation $\mathrm{c}_1\mathrm{c}_2 = \text{const.}$ and $\mathrm{s}_1\mathrm{s}_2 = \text{const.}$, respectively, there is no possibility to replace the appearing sums $\sum \mathrm{c}_i \left(\sum \mathrm{d}_i \mathrm{s}_i\right)^2$, $\left(\sum \mathrm{c}_i\right)^2 \frac{d}{dx} \sum \mathrm{d}_i \mathrm{s}_i$ and $\sum \mathrm{s}_i \left(\sum \mathrm{c}_i \mathrm{d}_i\right)^2$, $\left(\sum \mathrm{s}_i\right)^2 \frac{d}{dx} \sum \mathrm{c}_i \mathrm{d}_i$, respectively. Up to our knowledge there is no appropriate relation involving Jacobian elliptic functions that would simplify the equations similar to (26), so that the speed v_2 for which $\widetilde{\psi}$ is a solution of Eq. (17) cannot be determined.

the coefficients of the NLSE and the KPE. Constraints result from the condition that suitable solutions exist (cf. Eq. (13)) and from the PDC. As is obvious from the following Table I there are rather many NLWEEs that exhibit suitable elliptic solutions. Thus, it seems interesting to check whether they lead to superposition solutions by applying Eqs. (15) and (16).

ACKNOWLEDGMENTS

One of us (J. N.) gratefully acknowledges support by the German Science Foundation (DFG) (Graduate College 695 "Nonlinearities of optical material").

APPENDIX: SUPERPOSITION SOLUTIONS OF THE NONLINEAR CUBIC-QUINTIC SCHRDINGER EQUATION (NLCQSE)

Following the lines described in Sections 1. and 2. the NLCQSE

$$i\psi_t + \psi_{xx} - (q_1|\psi|^2 + q_2|\psi|^4)\psi = 0, \tag{45}$$

(q_1, q_2 real constants) can be solved by applying the transformation

$$\psi(x,t) = f(z)\exp[i(\lambda t + r(z))], \quad z = x - ct. \tag{46}$$

Separating real and imaginary parts, we obtain

$$q_1 f(z)^3 + q_2 f(z)^5 - f''(z) + f(z)(\lambda - cr'(z) + r'(z)^2) = 0, \tag{47}$$

$$f'(z)(c - 2r'(z)) - f(z)r''(z) = 0, \tag{48}$$

where the prime denotes differentiation with respect to z.

Equation (48) can be integrated to yield

$$r'(z) = \frac{c}{2} + \frac{C_1}{f(z)^2}, \tag{49}$$

with C_1 a constant of integration.

Inserting $r'(z)$ into Eq. (47) and integrating the resulting expression leads to an equation where $h = f^2$ can be introduced. Thus, we find a basic equation $R(h)$ (cf. Eq. (3), $f \to h$) with the following coefficients:

$$\alpha = \frac{4}{3}q_2, \quad \beta = \frac{1}{2}q_1, \quad \gamma = \frac{4\lambda - c^2}{6}, \quad \delta = 2\,C_2, \quad \epsilon = -4\,C_1^2, \tag{50}$$

where C_2 is a constant of integration.

If $q_2 = 0$ and $C_1 = 0$ all physical solutions suitable for superposition are represented by Eqs. (15) ($f \to h$). The superposition solutions for $p = 3$ are

Table I. Elliptic Solutions of Various Nonlinear Wave and Evolution Equations

Equation	Ansatz	Basic equation	Suitable for superposition
$\psi_t - \psi\psi_x - \psi_{xxt} = 0$ Benjamin-Bona-Mahony	$\psi = f(kx - ct) = f(z)$	$(f_z)^2 = -\frac{f^3}{3kc} + \frac{f^2}{k^2} + 4\delta f + \epsilon$	+
$\psi_{tt} - \psi_{xx} + 3(\psi^2)_{xx} - \psi_{xxxx} = 0$ Boussinesq	$\psi = f(kx - ct) = f(z)$	$(f_z)^2 = \frac{2f^3}{k^2} + \frac{c^2-k^2}{k^4} f^2 + 4\delta f + \epsilon$	+
$\psi_t + \psi\psi_x - \psi_{xx} = 0$ Burgers	$\psi = f(kx - ct) = f(z)$	$(f_z)^2 = \frac{(2c^2 f - kcf^2 + 4k^4\delta)^2}{4k^4c^2}$	–[a]
$\psi_{tt} - \psi_{xx} + \sin\psi + \frac{1}{2}\sin\frac{\psi}{2} = 0$ Double sine-Gordon	$\psi = 4\arctan[f(kx - ct)] = 4\arctan[f(z)]$	$(f_z)^2 = 3(\gamma + \frac{1}{8(c^2-k^2)}) f^4 + 6\gamma f^2 + 3\gamma + \frac{5}{(c^2-k^2)}, c^2 \neq k^2$	+
$\psi_t + a\psi_{xx} - b\psi - c\mid\psi\mid^2\psi = 0$ $a = a_1 + ia_2$, $b = b_1 + ib_2$, $c = c_1 + ic_2$ Ginzburg-Landau	$\psi = f(x)\exp(ig(x))\exp(i\lambda t)$, $g_x(x) = d\frac{f_x}{f}$	$(f_x)^2 = \frac{c_2}{3d\,a_1 + a_2(2-d^2)} f^4 + \left(\frac{\lambda_\pm - b_2}{d^2 a_2 - a_2 - 2d\,a_1}\right) f^{2b}$	+
$\psi_t + (b\psi + 1)\psi_x + \psi_{ttx} = 0$ Joseph-Egri Hereman *et al.* (1986)	$\psi(x, t) = f(kx - ct) = f(z)$	$(f_z)^2 = -\frac{b}{3c^2} f^3 + \frac{c-k}{c^2k} f^2 - \frac{2C_1}{c^2k} f - \frac{2C_2}{c^2k}$, C_1, C_2 const.	+
$\psi_t + a\psi^2\psi_x + b\psi_x\psi_{xx} + g\psi\psi_{xxx} + \psi_{xxxx} = 0$ Korteweg-de Vries	$\psi(x, t) = f(x - ct) = f(z)$	$(f_z)^2 = 4\beta f^3 + 6\gamma f^2 + 4\delta f + \epsilon$	+
$\psi_t - \psi_{xx} - 6\psi\psi_x + \psi_{xxx} = 0$	$\psi = f^2(kx - ct) = f^2(z)$	$(f_z)^2 = \frac{f^2}{k^2} \cdot \left(\frac{f^2}{2} \pm \frac{2f}{5\sqrt{2}} + \frac{1}{25}\right)$	–[c]

Table I. Continued

Equation	Ansatz	Basic equation	Suitable for superposition
Korteweg-de Vries-Burgers			
$\psi_t + a\psi\psi_x + \psi_{xxx}$ $+\psi_{xyy} + \psi_{xzz} = 0$ KdV-Zakharov-Kusnetzov Baldwin *et al*. (2004)	$\psi(x, y, z, t) = f(\xi)$ $\xi = kx + ly + mz + \omega t$	$(f_\xi)^2 = -\frac{a}{3p^2} f^3 - \frac{\omega}{kp^2} f^2 - \frac{2C_1}{kp^2}$, $p^2 = k^2 + l^2 + m^2$, C_1 const.	+
$V_t = \partial^3 V + \overline{\partial}^3 V$ $+3\partial(uV) + 3\overline{\partial}(\overline{u}V)$, $\overline{\partial}u = \partial V$ Novikov-Veselov	$V(x, y, t) = \psi(z)$ $z = x + ky - vt$	$(f_z)^2 = -\frac{8a_2}{1+k^2} f^3 - \frac{24a_0}{1+k^2} f^2 + \frac{12F}{3(3k^2-1)} f^2 + 4\delta f + \epsilon$, $F = v + 3C_0 + 3kC_1$; C_0, C_1 const.	+
$\psi_{xx} - \psi_{tt} - \sin\psi = 0$ sine-Gordon	$\psi(x, t) = 4\arctan\left[\frac{X(x)}{T(t)}\right]$	$\left(\frac{dX}{dx}\right)^2 = R_1(X) = \alpha X^4 + 6\gamma X^2 + \epsilon$ $(\frac{dT}{dt}) = \ldots$ (as below) $(\frac{dT}{dt})^2 = R_1(T) = \alpha T^4 + (6\gamma - 1)T^2 - \epsilon$	+
$i\psi_x + \psi_{tt} + 2\sigma\|\psi\|^2\psi$ $-\mu\psi_{xt} = 0$ Wadati *et al*. (1992)	$\psi(x, t) = f(z)e^{i(rx-\lambda t)}$ $z = kx - ct$	$(f_z)^2 = -\frac{\sigma}{c(c+k\mu)} f^4 - \frac{k(1+\lambda\mu)^2 + c\lambda(2+\lambda\mu)}{c^2\mu(c+k\mu)} f^2 - \frac{2C_1}{c(c+k\mu)}$, C_1 const.	+

Note. If condition (13) can be fulfilled (e.g., by choosing an appropriate constant of integration) start solutions for superposition can be obtained by Eqs. (15), (16) (marked by a "+", otherwise by a "−").

[a] As outlined above start solutions for linear superposition can be obtained if $\alpha = 0$ or $\beta = \delta = 0$. In the case of the Burgers equation these conditions lead to $k \to \infty$ or $c = 0$ for a traveling wave ansatz $\psi = f(kx - ct)$.

[b] d only depends on the coefficients of the Ginzburg-Landau equation a, b, c.

[c] As outlined above start solutions for linear superposition can be obtained if $\alpha = 0$ or $\beta = \delta = 0$. In the KdV-Burgers equation the only parameter which can be varied in the basic equation is k. If $k \to \infty$ then $\alpha \to 0$, but also $(f_z)^2 \to 0$ so that $f \equiv$ const.

given by (cf. Eqs. (2), (46))

$$\widetilde{\psi}(x,t) = a\sum_{i=1}^{3}\text{cn}\left[\mu(x - v_3t) + \frac{4(i-1)K(m)}{3}, m\right]\exp\left\{i\left[\lambda t + (x - v_3t)\frac{v_3}{2}\right]\right\},$$

$$v_3^2 = 4(\lambda - \mu^2(2mX(3,m) + (2m-1))), \tag{51}$$

$$\widetilde{\psi}(x,t) = a\sum_{i=1}^{3}\text{dn}\left[\mu(x - v_3t) + \frac{2(i-1)K(m)}{3}, m\right]\exp\left\{i\left[\lambda t + (x - v_3t)\frac{v_3}{2}\right]\right\},$$

$$v_3^2 = 4(\lambda + \mu^2(m-2) - aW(3,m)), \tag{52}$$

$$\widetilde{\psi}(x,t) = a\sum_{i=1}^{3}\text{sn}\left[\mu(x - v_3t) + \frac{4(i-1)K(m)}{3}, m\right]\exp\left\{i\left[\lambda t + (x - v_3t)\frac{v_3}{2}\right]\right\},$$

$$v_3^2 = 4(\lambda + \mu^2(m+1) + 2ma\mu^2V(3,m)). \tag{53}$$

To evaluate the speed v_3 we have used in Cooper *et al.* (2002) the Eqs. (8), (70), (72), Eqs. (8), (66), (68) and Eqs. (8), (57), (59), respectively.

It should be mentioned that the start solutions (15) suitable for superposition are consistent with those of Cooper *et al.* (2002). Nevertheless, the speed v_3 according to Eqs. (51), (52), (53) is not identical with v_3 according to Eqs. (33), (28), (45) in Cooper *et al.* (2002). Thus, the superposition solutions are not determined uniquely. Different identities between Jacobian elliptic functions used lead to (in general) different superposition solutions. Applying the procedure outlined in Section 2. if $q_2 \neq 0$ ($\alpha \neq 0$), $\beta = \delta = 0$, ϵ arbitrary, PDC implies either $q_2 = 0$ ($\alpha = 0$) or $C_1^2 = 0$ ($\epsilon = 0$). The choice $q_2 = 0$ (in addition to $q_1 = 0$ ($\beta = 0$)) is not of interest, because it leads to a linear Eq. (45). For $C_1^2 = 0$ we obtain solitary traveling-waves. Thus, since $\psi(x,t)$ is not periodic, superposition solutions are not possible in this case.

REFERENCES

Abramowitz, M. and Stegun, I. A. (1972). *Handbook of Mathematical Functions*, 9th ed., New York, Dover Publications, pp. 651–652.

Baldwin, D. *et al.* (2004). Symbolic computation of exact solutions expressible in hyperbolic and elliptic functions for nonlinear PDEs. *J. Symb. Comp.* **37**, 669–705.

Bronstein, I. N. *et al.* (2000). *Taschenbuch der Mathematik* 5th ed., Thun und Frankfurt am Main, Verlag Harri Deutsch, pp. 40–41.

Chandrasekharan, K. (1985). *Elliptic Functions*, Berlin, Springer, p. 44.

Cooper, F. *et al.* (2002). Periodic solutions of nonlinear equations obtained by linear superposition. *J. Phys. A: Math. Gen.* **35**, 10085–10100.

Drazin. P. G. (1983). *Solitons*, Cambridge, Cambridge University Press, p. 15.

Hereman, W. *et al*. (1986). Exact solitary wave solutions of nonlinear evolution and wave equations using a direct algebraic method. *J. Phys. A: Math. Gen.* **19**, 607–628.

Jaworski, M. and Lakshmanan, M. (2003). Comment on "Linear superposition in nonlinear equations." *Phys. Rev. Lett.* **90**, 239401–1.

Khare, A. and Sukhatme, U. (2002). Cyclic identities involving Jacobi elliptic functions. *J. Math. Phys.* **43**, 3798–3806.

Khare, A. and Sukhatme, U. (2002). Linear superposition in nonlinear equations. *Phys. Rev. Lett.* **88**, 244101-1–244101-4.

Khare, A. *et al*. (2003). Cyclic identities for Jacobi elliptic and related functions. *J. Math. Phys.* **44**, 1822–1841.

Schürmann, H. W. (1996). Traveling-wave solutions of the cubic-quintic Schrödinger equation. *Phys. Rev. E* **54**, 4312–4320.

Schürmann, H. W. and Serov, V. S. (2004). Traveling wave solutions of a generalized modified Kadomtsev-Petviashvili equation. *J. Math. Phys.* **45**, 2181–2817.

Schürmann, H. W. and Serov, V. S. (2004). Weierstrass' solutions to certain nonlinear wave and evolution equations. *Proc. Progress in Electromagnetics Research Symposium* March 28–31 (Pisa), pp. 651–654.

Wadati, M. *et al*. (1992). A new hamiltonian amplitude equation governing modulated wave instabilities. *J. Phys. Soc. Jpn.* **61**, 1187–1193.

Weierstrass, K. (1915). *Mathematische Werke V*, New York, Johnson, pp. 4–16, Whittaker, E. T. and Watson, G. N. (1927) *A Course of Modern Analysis*, Cambridge, Cambridge University Press, p. 454.